IIPV class of auctions

Identically distributed, Independent, Private Value

Assumptions

A1: Bidders are risk neutral

A2: Values are private and independently distributed

A3: Bidders are symmetric (ex-ante identical)

A4: Payments depend on bids alone (non-discrimination)

A5: Seller is risk-neutral

A6: There is a single, indivisible unit for sale

Postulate some trading institution (a mechanism) and look for BNE bidding strategies.

BNE - Bayesian Nash Equilibrium

In equilibrium, bidders submit optimal bids as functions of their valuations (types) and correct expectations about the bidding strategies of the other bidders.

English Auction (Open Ascending Bid)

In an OAB auction it is a weakly dominant strategy for bidder i with valuation v_i to remain in the bidding until the price, p, exceeds v_i.

Seller Revenue = $V_{n-1:n}$ or $V_{n-1:n} + \varepsilon$

Buyer Payoffs = $V_n - V_{n-1}$ to winner

= 0 to all losers

Hence,

Expected payoff to participating bidder

= Expected value of $V_n - V_{n-1}$

= {probability of winning} × {expected surplus when winning}

SPSB Auction

In an SPSB auction it is a weakly dominant strategy to bid one's value v_i.

- Bidding $b_i = v_i$ gives $v_i - \max_{j\neq i}\left[b_j\right]$ if $v_i > \max_{j\neq i}\left[b_j\right]$ and 0 otherwise.
- Bidding $b_i > v_i$ is suboptimal, since there is a strictly positive probability that $b_i > \max_{j\neq i}\left[b_j\right] > v_i$ in which case I is the winner with $v_i - \max_{j\neq i}\left[b_j\right] < 0$. (Winner's curse)
- Bidding $b_i < v_i$ is suboptimal, since there is a strictly positive probability that $v_i > \max_{j\neq i}\left[b_j\right] > b_i$ in which case $v_i - \max_{j\neq i}\left[b_j\right] > 0$ and i is not winner.

First equivalence result

The OAB and SPSB auctions are equivalent, in the sense that the optimal bidding strategies are essentially the same (in the IIPV case). Hence, they give rise to the same expected payments and revenues.

In OAB and SPSB auction the payment of the winner's payment is independent of his own bid.

Expected revenue for the seller

In OAB and SPSB auctions the expected revenue equals the expected value of the second-highest order statistics from the sample on the n valuations.

$$E\left[P\right] = E\left[V_{n-1:n}\right]$$

$$E(V_{[n-1]}) = \int_{\underline{v}}^{\overline{v}} v f_{n-1,n}(v) dv$$

$$= \int_{\underline{v}}^{\overline{v}} n\left(n-1\right) v f\left(v\right)\left(1-F\left(v\right)\right) F\left(v\right)^{n-2} dv$$

$$= \int_{\underline{v}}^{\overline{v}} vn(1 - F(v)) f_{n-1,n-1}(v)\ dv$$

Integrate by parts:

$$= \left[nv\left(1-F\left(v\right)\right)F\left(v\right)^{n-1}\right]_{\underline{v}}^{\overline{v}} - \int_{\underline{v}}^{\overline{v}} F\left(x\right)^{n-1} \frac{d}{dx}\left[nx\left(1-F\left(x\right)\right)\right] dx$$

$$= 0 + \int_{\underline{v}}^{\overline{v}} F\left(x\right)^{n-1} xnf\left(x\right) dx - \int_{\underline{v}}^{\overline{v}} F\left(x\right)^{n-1} \left[n\left(1-F\left(x\right)\right)\right] dx$$

$$= \int_{\underline{v}}^{\overline{v}} \left[v - \frac{1-F\left(v\right)}{f\left(v\right)}\right] f_{n,n}\left(v\right) dv$$

Interpretation:

$$MR\left(v\right) = v - \frac{1-F\left(v\right)}{f\left(v\right)}$$

Is defined in the literature as the marginal revenue to the seller from a bidder of type v. The expected revenue from the auction is therefore the expected marginal revenue of the highest bidder (the bidder with the highest valuation).

The expected price for a bidder with valuation v_i is equal to the expected value of the highest of $(n-1)$ draws from 0 to v_i :

$$P(v_i)=\int_{\underline{v}}^{v_i} w dF(w)^{n-1} = v_i F(v_i)^{n-1} - \int_{\underline{v}}^{v_i} F(w)^{n-1} dw$$

The expected price to be paid conditional upon winning with v_i is :

$$E\left[P(v_i)\middle| Win\right] = \frac{P(v_i)}{F(v_i)^{n-1}}$$

The expected surplus from winning is therefore:

$$\begin{aligned} E\left[\Pi_i\right] &= (v_i - P)F(v_i)^{n-1} \\ &= v_i F(v_i)^{n-1} - E(P) \\ &= \int_{\underline{v}}^{v_i} F(w)^{n-1} dw \end{aligned}$$

Symmetric equilibrium of FPSB auction (and Dutch)

Second equivalence result

Formally, i.e. from a game theory perspective, the Dutch and the FPSB auctions are equivalent. Thus, bidding strategies and payoffs should be identical.

In both formats bidders have to make bids without learning anything from the choices of others. The two formats have the same reduced normal form (Strategic Equivalence). Heuristically, whether I sit in an auction room waiting for the price to fall to my strike price (bid) or whether I leave my bid (strike price) in a sealed envelope is inconsequential.

Assume that there exist a symmetric and monotone increasing function which represents the equilibrium of the FPSB auction: $\beta(\cdot)$. Function $\beta(\cdot)$ generates a bid b for a given valuation v. Bidder i observes v_i and he assumes the other $N-1$ valuations as (independent) random draws from $F(\cdot)$. These can be written (in ordered form) as:

$$Y_{(N-1:N-1)} > Y_{(N-2:N-1)} > \ldots > Y_{(N:N-1)}$$

Now if the other $N-1$ bidders follows $\beta(\cdot)$, what is the optimal reply for bidder i?

Expected surplus of bidder i when the others follow $\beta(\cdot)$: $(v_i - b_i)\Pr[i\ wins]$ or 0 otherwise. The probability of winning under $\beta(\cdot)$ is the probability of the event:

$$b_i > b_{(N-1:N-1)} = \beta\left(Y_{(N-1:N-1)}\right)$$

For β increasing wrt *v*. Expected profit of bidder i is then given by:

$$E\left[\Pi_i\right] = \left(v_i - b_i\right) F_{(N-1:N-1)}\left(\sigma\left(b_i\right)\right) \tag{1}$$

where $\sigma(b_i)$, is the inverse bid function, namely that value of v_i such that in the equilibrium $\beta(\cdot)$ generates a bid b_i. Max (1) w.r.t. b_i, generates the following FOC:

$$\left(v_i - b_i\right) f_{(N-1:N-1)}\left(\sigma\left(b_i\right)\right)\frac{\partial \sigma}{\partial b_i} - F_{(N-1:N-1)}\left(\sigma\left(b_i\right)\right) = 0$$

Notice that given the assumption about $\beta(v_i)$:

$$\sigma\left(b_i\right) = v_i \text{ and } \frac{\partial \sigma}{\partial b_i} = \frac{1}{\beta'}.$$

We rewrite the FOC as:

$$\begin{aligned}&\beta'\left(v_i\right) F_{(N-1:N-1)}\left(v_i\right) + \beta\left(v_i\right) f_{(N-1:N-1)}\left(v_i\right) = \\ &= v_i f_{(N-1:N-1)}\left(v_i\right)\end{aligned}$$

Given our definitions of $F_{(N-1:N-1)}(v_i)$ and $f_{(N-1:N-1)}(v_i)$:

$$\beta'\left(v_i\right) + \beta\left(v_i\right)\frac{(N-1) f\left(v_i\right)}{F\left(v_i\right)} = v_i \frac{(N-1) f\left(v_i\right)}{F\left(v_i\right)} \tag{2}$$

rewrite (2) as:

$$\frac{d}{dv_i}\left[\beta\left(v_i\right) F_{(N-1:N-1)}\left(v_i\right)\right] = v_i f_{(N-1:N-1)}\left(v_i\right)$$

Using the initial condition $\beta(0) = 0$, we solve the above differential equation as:

$$\beta\left(v_i\right) = \int_{\underline{v}}^{v_i} \tilde{v} \frac{f_{(N-1:N-1)}\left(\tilde{v}\right)}{F_{(N-1:N-1)}\left(v_i\right)} d\tilde{v} \tag{3}$$

or:

$$b_i = \frac{\int_{\underline{v}}^{v_i} \tilde{v}(N-1) f\left(\tilde{v}\right) d\tilde{v}}{F\left(v_i\right)^{N-1}}$$

It is possible to show that (3) is indeed a sym eqm of FPSB auction.

Bid function (3) is equivalent to:

$$E\left[V_{(N-1:N-1)} \middle| V_{(N-1:N-1)} < v_i\right]$$

Given that

$$\frac{f_{(N-1:N-1)}\left(\tilde{v}\right)}{F_{(N-1:N-1)}\left(v_i\right)} = f_{(N-1:N-1)}\left(\tilde{v} \middle| \tilde{v} < v_i\right) \tag{4}$$

the optimal bid is increasing in N since (4) is increasing in N and the integral of (3) has a positive value. In the same manner the expected selling price in FPSB auction is increasing in N.

Next rewrite the bidding function as:

$$\beta(v_i) = v_i - \frac{\int_{\underline{v}}^{v_i} F^{N-1}(\tilde{v})\, d\tilde{v}}{F^{N-1}(v_i)}$$

From this it is immediate to notice that the bid is a "mark down" of the true valuation v_i. We assumed above that $\beta(.)$ was strictly increasing, and we used this when we inverted the bidding function. Can we show that the bidding function is, indeed, increasing?

$$\begin{aligned}\frac{d\beta}{dv_i} &= 1 - \frac{F^{\,n-1}(v_i)F^{\,n-1}(v_i) - (n-1)f(v_i)F^{\,n-2}(v_i)\int_{\underline{v}}^{v_i} F^{\,n-1}(x)dx}{\left[F^{\,n-1}(v_i)\right]^2} \\ &= \frac{(n-1)f(v_i)F^{\,n-2}(v_i)}{\left[F^{\,n-1}(v_i)\right]^2}\int_{\underline{v}}^{v_i} F^{\,n-1}(x)dx > 0\end{aligned} \qquad (5)$$

Thus, $\beta(\cdot)$is strictly increasing. Since $\beta(\cdot)$is strictly increasing, the high-valuation bidder (with $v = V_{[1]}$) wins, and we conclude that the auction is efficient (in the sense that it puts the item in the hands of the bidder who values it the most).

INTRODUCTORY EXAMPLE

The simplest case is when the distribution of bidders' valuations is uniform:

$$\tilde{v} \sim U[0,1]$$

More generally, the uniform distribution on $[a,b]$ is:

$$F(v) = \frac{v-a}{b-a}$$

Assume that there are just two bidders i and j and j is known to follow a bidding strategy $b_j = \beta(v_j)$.

Then bidder i wins if:

$$b_i > b_j$$

$$b_i > \beta(v_j)$$

$$v_j < \sigma(b_i)$$

Where $\sigma = \beta^{-1}$ is the inverse bidding function. Then the probability of winning in the Uniform case is simply $F(\sigma(b_i)) = \sigma(b_i)$.

If there are N bidders with IID valuations the probability of winning is: $\left(\sigma(b_i)\right)^{N-1}$.

Bidder i maximizes the expected profit form the auction:

$$\max_b E[\Pi_i] = (v_i - b_i)\left(\sigma(b_i)\right)^{N-1}$$

FOC

$$-\left(\sigma(b_i)\right)^{N-1} + (v_i - b_i)(N-1)\left(\sigma(b_i)\right)^{N-2}\sigma'(b_i) = 0$$

In equilibrium $\sigma(b_i) = v_i$ and $\sigma'(b_i) = \frac{1}{\beta'(v_i)}$, then:

$$-\beta'(v_i)(v_i)^{N-1} + (v_i - b_i)(N-1)(v_i)^{N-2} = 0$$

$$\beta'(v_i)(v_i)^{N-1} + b_i(N-1)(v_i)^{N-2} = v_i(N-1)(v_i)^{N-2}$$

$$\frac{d(\beta(v_i) \cdot (v_i)^{N-1})}{dv_i} = v_i(N-1)(v_i)^{N-2}$$

$$\beta(v_i) \cdot (v_i)^{N-1} = const + \int_0^{v_i} x(N-1)(x)^{N-2} dx$$

$$const = \beta(\underline{v})(\underline{v})^{N-1} = 0$$

$$\beta(v_i) = \frac{\int_0^{v_i} x(N-1)(x)^{N-2} dx}{(v_i)^{N-1}}$$

This can be read as the expected value of the second highest valuation among $(N-1)$ given that it is less than v_i. Solve as follows:

$$b_i = \frac{(N-1)}{N} \frac{(v_i)^N}{(v_i)^{N-1}} = \frac{(N-1)}{N} v_i$$